On the Conductivity of a Gas, between Parallel Plate Electrodes, when the Current Approaches the Maximum Value.

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The relation connecting the current with the potential difference between parallel plate electrodes when the gas between the plates has been uniformly ionised by Röntgen rays or Becquerel rays has been investigated theoretically by many physicists. In all cases various assumptions are made in order to simplify the calculations, as the problem becomes very complicated when the disturbance of the field due to the separation of the ions is taken into consideration.

Perhaps the most complete solution is that given by Mie,* in which the only effect that is neglected is that of diffusion. The difference between the velocities of the positive and negative ions is taken into consideration, and the disturbance of the field due to the charge in the gas produced by the excess of ions of one sign in the neighbourhood of the electrodes. The method of analysis, consisting of a series of approximations, is difficult, but the results have been presented in a convenient form, for currents in air at atmospheric pressure that are certain fractions of the saturation current. A curve is given for each current which shows the distribution of force between the plates. The currents investigated ranged between those that were one-fifth and nine-tenths of the saturation current. In the former case the ratio of the electric force at the negative electrode to the minimum force in the field was found to be 2·7. The ratio diminishes as the force increases, and for the current that is nine-tenths of the saturation current the ratio becomes 1·39.

Another investigation of the field of force between parallel plate electrodes when a current is flowing is given by Sir J. J. Thomson,† in which equal velocities are attributed to positive and negative ions. The conclusion arrived at is that the ratio of the maximum to the minimum force is 2·51, a constant quantity, independent of the current or of the intensity of ionisation. The number 2·51 corresponds to air at atmospheric pressure. For lower pressures the ratio is said to increase and to vary inversely as the square root of the pressure. These conclusions are unsatisfactory, as it can easily be shown that the ratio of the maximum to

* Mie, 'Ann. der Physik,' 1904, vol. 13, p. 857.

† 'Conduction of Electricity through Gases,' pp. 84—87.

the minimum force is not constant but diminishes as the rate of ionisation is reduced, or as the force increases. Also for a given force and intensity of ionisation the polarisation of the gas must diminish as the pressure is reduced.

The method adopted by Mie is necessarily very complicated, as it applies to cases in which there are large variations in the field of force, and as it is necessary for many purposes to know the degree of saturation corresponding to a given electromotive force the following simple investigation of the conductivity of a gas between parallel plates when high forces are used may be of interest.

It is easy to see that when the potential difference between the electrodes is sufficiently large the polarisation may be neglected and the field of force between parallel plates may be considered uniform. The conductivity in a uniform field is therefore of a kind that can be realised in practice, and from the solution of the equations obtained for that case it can be shown how the charge in the gas increases as the force is reduced, and it is possible to find exactly the percentage by which the current falls below the maximum current before the field of force is disturbed to such an extent as to introduce a serious error in the determination of the current.

In order to find an upper limit to the difference between the maximum force X_1 to the minimum force X_2 in a conducting gas, it is necessary to find the charge in the gas between the two points at which the forces are acting. Let ν_1 and ν_2 be the numbers of positive and negative ions per cubic centimetre when the steady state is reached and a current i is flowing. If e denotes the charge on an ion the current i per unit area of the electrodes is $e(\nu_1 u_1 + \nu_2 u_2)$ where u_1 and u_2 are the velocities of the positive and negative ions. Letting $n_1 = e\nu_1$ and $n_2 = e\nu_2$, then

$$i = n_1 u_1 + n_2 u_2.$$

The velocities of the ions are proportional to the electric force, so that

$$u_1 = k_1 X \quad \text{and} \quad u_2 = k_2 X.$$

Also let a be the distance between the electrodes. The difference between the greatest and smallest force is

$$X_1 - X_2 = 4\pi \int (n_1 - n_2)\, dx = 4\pi \int \left[\frac{n_1 u_1 + n_2 u_2}{u_1} - \frac{n_2 (u_1 + u_2)}{u_1} \right] dx.$$

Hence
$$X_1 - X_2 < 4\pi \int \frac{i}{u_1}\, dx,$$

where the integral is taken over the distance between the two points at which the forces are acting, which is less than the distance a.

Hence, if u_1 is the least velocity in the field,

$$X_1 - X_2 < \frac{4\pi a i}{u_1'}, \quad \text{so that} \quad \frac{X_1 - X_2}{X_2} < \frac{4\pi a i}{k_1 X_2^2}.$$

The force X_2 may be increased to a large value, but the current i remains constant when the saturation point is reached, and the quantity $(X_1 - X_2)/X_2$ becomes so small that the force may be considered constant.

Let the space between the plates be ionised uniformly by rays producing q/e positive or negative ions per cubic centimetre in unit time, and let the rate of recombination be αe, so that the number of ions of either kind that recombine per second is $\alpha e \nu_1 \nu_2$.

When the steady state is reached the rate at which ions are being generated per unit volume must be equal to the loss by recombination, together with the loss by the outward motion of ions through the boundary of the volume. When the diffusion of the ions is neglected the steady state is represented by the equation

$$\frac{q}{e} = \alpha e \nu_1 \nu_2 + \frac{d}{dx}(\nu_1 u_1),$$

or

$$\frac{d}{dx}(n_1 u_1) = q - \alpha n_1 n_2. \tag{1}$$

Similarly

$$-\frac{d}{dx}(n_2 u_2) = q - \alpha n_1 n_2. \tag{2}$$

Eliminating n_2 from the above equation, the following differential equation for $n_1 u_1$ is obtained:—

$$\frac{d}{dx}\left\{\frac{u_2}{\alpha n_1}\left(q - \frac{d}{dx}(n_1 u_1)\right)\right\} = -\frac{d}{dx}(n_1 u_1),$$

which on integration gives

$$\frac{d}{dx}(n_1 u_1) = \frac{\alpha}{u_1 u_2}\left[n_1^2 u_1^2 + C n_1 u_1 + q\frac{u_1 u_2}{\alpha}\right].$$

The constant of integration C is equal to $-i$, where $i = n_1 u_1 + n_2 u_2$ is the current per unit area between the plates.

On integration this equation gives

$$\frac{1}{\sqrt{(q u_1 u_2/\alpha - \frac{1}{4} i^2)}} \tan^{-1}\left(\frac{n_1 u_1 - \frac{1}{2} i}{\sqrt{(q u_1 u_2/\alpha - \frac{1}{4} i^2)}}\right) = \frac{\alpha x}{u_1 u_2} + C_1,$$

or

$$n_1 u_1 - \tfrac{1}{2} i = \sqrt{(q u_1 u_2/\alpha - \tfrac{1}{4} i^2)} \tan\left(\frac{\alpha}{u_1 u_2}(x - \tfrac{1}{2} a)\sqrt{(q u_1 u_2/\alpha - \tfrac{1}{4} i^2)}\right), \tag{3}$$

the constant C_1 being determined by the conditions $n_1 u_1 = i$ when $x = a$, and

$n_1 u_1 = 0$ when $x = 0$. The equation (4) connecting the current i and the electric force X is also obtained from the latter condition

$$\sqrt{\left(\frac{4qk_1k_2\mathrm{X}^2}{\alpha i^2}-1\right)}\tan\left[\frac{\alpha a i}{4k_1k_2\mathrm{X}^2}\sqrt{\left(\frac{4qk_1k_2\mathrm{X}^2}{\alpha i^2}-1\right)}\right] = 1. \tag{4}$$

In order to see over what range of forces this equation may be considered to hold accurately, it is necessary to find to what extent the field becomes disturbed by the charge $n_1 - n_2$ per cubic centimetre of the gas. For this purpose it will be sufficient to consider that the ions move with equal velocities, so that there will be the same increase of force at each electrode, and the minimum force will be at the centre of the field.

Let $p^2 = 4qk_1k_2\mathrm{X}^2/\alpha i^2$, then equation (3) becomes, on substituting for i its equivalent $n_1u_1 + n_2u_2$,

$$n_1u_1 - n_2u_2 = i\sqrt{(p^2-1)}\tan\left\{\frac{2q}{i}\frac{x-\frac{1}{2}a}{p^2}\sqrt{(p^2-1)}\right\},$$

in which u_1 and u_2 may be taken as equal, and a mean value of the difference between the forces at the electrodes X_1 and the minimum force X_2 will be given by the equation

$$\mathrm{X}_1 - \mathrm{X}_2 = 4\pi\int_0^{a/2}(n_1-n_2)\,dx = -\frac{4\pi i^2 p^2}{2qu}\log\cos\left\{\frac{aq}{i}\frac{\sqrt{(p^2-1)}}{p^2}\right\},$$

or

$$\frac{\mathrm{X}_1-\mathrm{X}_2}{\mathrm{X}} = -\frac{8\pi k_1k_2}{\alpha k}\log\cos\left\{\frac{aq}{i}\frac{\sqrt{(p^2-1)}}{p^2}\right\}, \tag{5}$$

where k is the mean velocity of an ion under unit electric force.

As an example of the application of equations (4) and (5) the conductivity of air at atmospheric pressure may be considered. In this case the mean velocity of the ions may be taken as 450 cm. per second under a force of one electrostatic unit, and $\alpha = 3400$, so that equation (5) becomes

$$\frac{\mathrm{X}_1-\mathrm{X}_2}{\mathrm{X}} = -3{\cdot}3\log\cos\phi, \tag{5a}$$

where

$$\phi = \frac{aq}{i}\frac{\sqrt{(p^2-1)}}{p^2}. \tag{6}$$

Since the investigation applies only to cases in which the ratio $(\mathrm{X}_1-\mathrm{X}_2)/\mathrm{X}$ is small, equation (5) a shows that ϕ must be a small angle, so that from equation (4), which reduces to $p^2-1 = \phi^{-2}-\frac{2}{3}$, equation (6) gives

$$\frac{i}{aq} = \frac{1}{1+\frac{2}{3}\phi^2}. \tag{6a}$$

This shows that the current must be nearly saturated, since aq is the maximum value of i.

The current in terms of the electric force is given by the equation

$$\frac{4qk_1k_2X^2}{\alpha i^2} = \frac{1}{\phi^2}+1 = \frac{1}{\phi^2}, \tag{7}$$

since ϕ is a small quantity.

On substituting the values of the constants for air at atmospheric pressure the following relation is obtained:—

$$X = \frac{\sqrt{(ai)}}{15\cdot5} \times \frac{1}{\phi}. \tag{7a}$$

This equation, combined with the equations

$$\frac{i}{aq} = \frac{1}{1+\frac{2}{3}\phi^2}, \quad \text{and} \quad \frac{X_1-X_2}{X} = \frac{3\cdot3\,\phi^2}{2},$$

gives the corresponding variables in terms of the parameter ϕ. Thus, giving ϕ the values 0·1, 0·141, and 0·3, the following relations hold for air at atmospheric pressure:—

$$\phi = 0\cdot1 \;\ldots\ldots\ldots\; X = \frac{\sqrt{(ai)}}{1\cdot55}; \quad i = \frac{aq}{1\cdot006}; \quad \frac{X_1-X_2}{X} = 0\cdot016,$$

$$\phi = 0\cdot141 \;\ldots\ldots\; X = \frac{\sqrt{(ai)}}{2\cdot17}; \quad i = \frac{aq}{1\cdot012}; \quad \frac{X_1-X_2}{X} = 0\cdot033,$$

$$\phi = 0\cdot3 \;\ldots\ldots\ldots\; X = \frac{\sqrt{(ai)}}{4\cdot65}; \quad i = \frac{aq}{1\cdot06}; \quad \frac{X_1-X_2}{X} = 0\cdot148.$$

The first row of figures show that in order to get a current that is 0·6 per cent. less than the maximum the mean electric force is given by the equation $X = \sqrt{(ai)}/1\cdot55$, and that there is a variation of 1·6 per cent. in the electric force in the field between the plates. Thus if the distance between the plates a be 1 cm. and the current $q = 10^{-3}$ in electrostatic units the potential difference between the plates is 6·1 volts.

In order to see to what extent the method is accurate, when applied to currents several per cent. less than the maximum current, the figures in the third row may be considered. In that case the force at the electrodes is greater by 15 per cent. than the force at a point midway between the plates where the rate of recombination is greatest, since the product n_1n_2 has a maximum value in that region. The length of time the ions are in the gas, where they recombine most rapidly, is therefore about 7 per cent. longer than that allowed for, and the numbers of positive and negative ions present are also greater than the computed numbers by the same percentage, owing to the reduction in the velocity of the ions. Consequently, the amount of recombination is underestimated to the extent of about 14 per cent., so that the current i should be $aq/1\cdot07$ instead of $aq/1\cdot06$.

The effect of reducing the pressure may be found by substituting for the

numerical constants their values for low pressures. The velocities k_1 and k_2 vary inversely as the pressure, and according to the most reliable determinations of the variation of the rate of recombination with pressure, which have been made by Langevin, the quantity α varies approximately in direct proportion to the pressure. If the pressure be reduced to the nth part of an atmosphere the equations for determining the variable quantities become

$$X = \frac{\sqrt{(ai)}}{15\cdot 5} \times \frac{1}{n^{3/2}\phi}; \quad \frac{i}{aq} = \frac{1}{1+\frac{2}{3}\phi^2}; \quad \frac{X_1 - X_2}{X} = \frac{3\cdot 3 n^2 \phi^2}{2}.$$

So that for a pressure of 1/100th atmosphere, when the current and electric force are the same as at one atmosphere, these equations give, on substituting for ϕ the value 10^{-4},

$$X = \frac{\sqrt{(ai)}}{1\cdot 55}; \quad \frac{i}{aq} = \frac{1}{1+6\times 10^{-9}}; \quad \frac{X_1 - X_2}{X} = 0\cdot 00016.$$

So that the effect of polarisation is reduced to 1/100th part of its original value, and the current practically attains its saturation value. In practice, however, the current would not be so easily saturated, for the effect of diffusion increases as the pressure is reduced in the same proportion as the velocities of the ions, and at low pressures the effect of recombination in reducing the current is generally of less importance than the effect of diffusion, which causes some of the ions to be lost by coming into contact with the electrode opposite to that towards which they tend to move by the action of the electric force.